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A calculation of SO(8) Clebsch–Gordan coefficients

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Abstract. All Clebsch–Gordan coefficients required for calculations in the meson sector of a proposed SO(8) model of elementary particles are obtained (as SU(4) singlet factors) by an extension of the general formalism of Gel'fand.

1. Introduction

In the past three years there has been a renewed interest in the possibility of new quantum numbers, and higher symmetry groups, in hadron physics. In particular, the narrow ψ resonances (Aubert *et al* 1974, Augustin *et al* 1974) have been associated with a new additive quantum number, 'charm' in an SU(4) scheme, and mesons carrying the new quantum number have subsequently been found (Goldhaber *et al* 1976, Peruzzi *et al* 1976). A new heavy τ^- lepton, thought to be associated with the charm threshold, has also been reported (Perl *et al* 1975). Also a growing body of evidence, not least the recent discovery of the Υ resonances (Herb *et al* 1977), suggests that there may be yet further hadronic degrees of freedom beyond charm, still to be elucidated.

One such scheme, incorporating a richer hadron spectrum, has a phenomenology based on the special orthogonal group SO(8) (Barnes *et al* 1978). The purpose of this paper is to provide the Clebsch–Gordan coefficients necessary to perform calculations within the SO(8) scheme for scattering and decay amplitudes in the meson sector.

In § 2, we summarise some important facts about SO(8). The computational technique for calculating the coefficients is described in § 3. Section 4 is a guide to tables 4 and 5, which give the singlet factors.

2. General properties of SO(8)

SO(8) is the real compact Lie group corresponding to the Lie algebra D_4. It has a natural subgroup SO(6) whose universal covering group is SU(4). In terms of Lie algebras, we have $D_4 \supset D_3 \cong A_3$. In this sense, SO(8) is a rank-four generalisation of the group SU(4).

This statement is made more precise by specifying the generators of the subgroups involved. We work with a set of 28 anti-Hermitian SO(8) generators

$$\Sigma_{AB} = -\Sigma_{BA}, \qquad A, B = 1, 2, \ldots, 8,$$

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with the commutation relations

$$[\Sigma_{AB}, \Sigma_{CD}] = \delta_{BC}\Sigma_{AB} + \delta_{AD}\Sigma_{BC} - \delta_{AC}\Sigma_{BD} - \delta_{BD}\Sigma_{AC}; \tag{2.1}$$

the 15 generators of SO(6) are the subset

$$\Sigma_{IJ} = -\Sigma_{JI}, \qquad I, J = 1, 2, \ldots, 6.$$

For the 15 generators of SU(4), we take the set†

$$A_{ab} = A^{\dagger}_{ba}, \qquad a, b = 1, 2, 3, 4,$$

with commutation relations

$$[A_{ab}, A_{cd}] = \delta_{bc}A_{ad} - \delta_{ad}A_{cb} \qquad \text{and} \qquad \sum_{a=1}^{4} A_{aa} = 0.$$

The Lie algebra isomorphism is established by constructing a non-singular mapping between the two sets of generators. Since such a mapping preserves the Lie bracket structure, it is sufficient to specify it for the Cartan subalgebras, and for a set of simple roots.

For SU(4) we choose as the Cartan subalgebra the set

$$H_a = A_{aa}, \qquad a = 1, 2, 3, 4.$$

The remaining generators have the root structure

$$r(A_{ab}) = e_a - e_b, \qquad a \neq b, \quad a, b = 1, 2, 3, 4.$$

For SO(8), we choose as the Cartan subalgebra the set

$$K_a = \mathrm{i}\,\Sigma_{2a-1\,2a}, \qquad a = 1, 2, 3, 4$$

and defining (for $a < b$)

$$\begin{aligned}
D_{ab} &= \tfrac{1}{2}(-\mathrm{i}\,\Sigma_{2a-1\,2b-1} + \Sigma_{2a-1\,2b} - \Sigma_{2a\,2b-1} - \mathrm{i}\,\Sigma_{2a\,2b} \\
D_{a\bar{b}} &= \tfrac{1}{2}(\mathrm{i}\,\Sigma_{2a-1\,2b-1} + \Sigma_{2a-1\,2b} + \Sigma_{2a\,2b-1} - \mathrm{i}\,\Sigma_{2a\,2b}) \\
D_{\bar{a}b} &= \tfrac{1}{2}(\mathrm{i}\,\Sigma_{2a-1\,2b-1} - \Sigma_{2a-1\,2b} - \Sigma_{2a\,2b-1} - \mathrm{i}\,\Sigma_{2a\,2b}) \\
D_{\bar{a}b} &= \tfrac{1}{2}(-\mathrm{i}\,\Sigma_{2a-1\,2b-1} - \Sigma_{2a-1\,2b} + \Sigma_{2a\,2b-1} - \mathrm{i}\,\Sigma_{2a\,2b})
\end{aligned} \tag{2.2}$$

we find the root structure

$$r(D_{ab}) = e_a - e_b \qquad r(D_{a\bar{b}}) = e_a + e_b$$

$$r(D_{\bar{b}}) = -e_a - e_b \qquad r(D_{\bar{a}\bar{b}}) = -e_a + e_b.$$

We can now write down the isomorphism of the Lie algebras of SU(4) and SO(6). We identify

$$\begin{aligned}
&A_{12} = D_{12} \qquad A_{23} = D_{23} \qquad A_{34} = D_{\bar{1}\bar{2}} \\
&H_i = K_i - H_4, \quad i = 1, 2, 3 \qquad H_4 = \tfrac{1}{3}(K_1 + K_2 + K_3).
\end{aligned} \tag{2.3}$$

Irreducible representations of SO(8) can be given explicitly in terms of the Gel'fand basis (Gel'fand and Zetlin 1950, Wong 1967, Gilmore 1970), which uses the

† Our notational conventions for indices are as follows:

$$i, j = 1, 2, 3; \qquad I, J = 1, 2, \ldots, 6; \qquad a, b = 1, 2, 3, 4; \qquad A, B = 1, 2, \ldots, 8.$$

property that every basis state in each irreducible representation of SO(8) is contained in precisely one sequence of irreducible representations in a descending subgroup chain,

$$\mathrm{SO}(8) \supset \mathrm{SO}(7) \supset \mathrm{SO}(6) \supset \ldots \supset \mathrm{SO}(3) \supset \mathrm{SO}(2). \tag{2.4}$$

Thus, states are specified by Gel'fand patterns,

$$\left| \begin{array}{ccccccc} \sigma_{18} & & \sigma_{28} & & \sigma_{38} & & \sigma_{48} \\ & \sigma_{17} & & \sigma_{27} & & \sigma_{37} & \\ & \sigma_{16} & & \sigma_{26} & & \sigma_{36} & \\ & & \sigma_{15} & & \sigma_{25} & & \\ & & \sigma_{14} & & \sigma_{24} & & \\ & & & \sigma_{13} & & & \\ & & & \sigma_{12} & & & \end{array} \right\rangle \tag{2.5}$$

where the σ_{aA} are all integers or all half-odd integers, and $[\sigma_{1A}, \sigma_{2A}, \ldots]$ specifies an irreducible representation of SO(A) such that

$$\begin{gathered} \sigma_{1\,2a+1} \geqslant \sigma_{1\,2a} \geqslant \sigma_{2\,2a+1} \geqslant \ldots \geqslant \sigma_{a-1\,2a+1} \geqslant \sigma_{a-1\,2a} \geqslant \sigma_{a\,2a+1} \geqslant \sigma_{a\,2a} \geqslant -\sigma_{a\,2a+1} \\ \text{and} \\ \sigma_{1\,2a} \geqslant \sigma_{1\,2a-1} \geqslant \sigma_{2\,2a} \geqslant \ldots \geqslant \sigma_{a-1\,2a} \geqslant \sigma_{a-1\,2a-1} \geqslant |\sigma_{a-1\,2a}| \geqslant 0 \end{gathered} \tag{2.6}$$

for $2a, 2a+1 = 2, 3, \ldots, 8$.

In the following, the SO(6) subgroup plays a central role. It is convenient to label states according to the modified subgroup chain

$$\mathrm{SO}(8) \supset \mathrm{SO}(6) \times \mathrm{SO}(2) \cong \mathrm{SU}(4)/Z_2 \times \mathrm{U}(1) \supset \mathrm{SU}(3) \supset \mathrm{SU}(2) \supset \mathrm{U}(1) \tag{2.7}$$

in order to emphasise this. The last three subgroups embody the more familiar Gel'fand labelling for SU(4) (Nagel and Moshinsky 1965, Haacke *et al* 1976) in which the diagonal quantum numbers are the component along the isospin quantisation axis, I_3, the 'strange' hypercharge, Y, and the 'charm' hypercharge, Z, defined by (Haacke *et al* 1976)

$$\begin{aligned} I_3 &= \tfrac{1}{2}\mathrm{i}(\Sigma_{12} - \Sigma_{34}) = \tfrac{1}{2}(H_1 - H_2) \\ Y &= \tfrac{1}{3}\mathrm{i}(\Sigma_{12} + \Sigma_{34} - 2\,\Sigma_{56}) = \tfrac{1}{3}(H_1 + H_2 - 2H_3) \\ Z &= -\tfrac{1}{2}\mathrm{i}(\Sigma_{12} + \Sigma_{34} + \Sigma_{56}) = -H_4 \end{aligned} \tag{2.8}$$

In addition, in this modified subgroup labelling chain, multiplets of $\mathrm{SO}(6) \cong \mathrm{SU}(4)/Z_2$ within an irreducible representation of SO(8) are distinguished by a further, additive, hypercharge-like quantum number X, defined by

$$X = \mathrm{i}\,\Sigma_{78} = K_4. \tag{2.9}$$

Hereafter, we shall refer to the basis for irreducible representations of SO(8), corresponding to the modified subgroup chain (equation (2.7)) loosely as the 'SU(4)× U(1)' basis. The basis transformation from the previous SO(7) ⊃ SO(6) basis is given in § 3.

It should be pointed out that the new basis does not provide a complete state labelling scheme, since we have replaced a set of three non-additive SO(7) labels with a single additive quantum number, X. For completeness, the basis should be augmented by the eigenvalues of two additional higher-order SO(6) invariants which commute with X, such as

$$
\begin{aligned}
X' &= \sum_{I,J=1}^{6} (\Sigma_{7I}\,\Sigma_{IJ}\,\Sigma_{J8} - \Sigma_{8I}\,\Sigma_{IJ}\,\Sigma_{J7}),\\
X'' &= \sum_{I,J=1}^{6} [\Sigma_{7I}(\Sigma^3)_{IJ}\Sigma_{J8} - \Sigma_{8I}(\Sigma^3)_{IJ}\Sigma_{J7}],
\end{aligned}
\tag{2.10}
$$

where

$$\Sigma_{IJ}^{n+1} = \sum_{K=1}^{6} (\Sigma^n)_{IK}\Sigma_{KJ}.$$

However, for the SO(8) multiplets which we are concerned with, these additional labels are redundant, and will not be considered further†.

The first few irreducible representations of SO(8) are listed in table 1, with their dimensions and SU(4)×U(1) restrictions. There are three eight-dimensional irreducible representations: two spinor ones, 8_+ and 8_-, and the defining (or vector) representations 8. Since $8_+ \times 8$ contains 8_-, it follows that only two of these three are elementary. Each irreducible representation of SO(8) is self-conjugate and orthogonal (and so can be brought to real form). The D_4 Dynkin diagram possesses a permutation symmetry associated with a group of outer automorphisms of D_4. The existence of this group leads to the occurrence of inequivalent irreducible representations of the same dimensions whose Dynkin labels $(a_1\, a_2\, {}^{a_3}_{a_4})$ (the components of the highest weight in the direction of the simple roots) differ only by a permutation of the three outer values. Thus, for example, there is only one representation $(0\,2\,{}^0_0)$, with dimensions 300, but three inequivalent representations $(2\,0\,{}^0_0)$, $(0\,0\,{}^2_0)$, $(0\,0\,{}^0_2)$, of the same dimension, called here 35, 35_+ and 35_-, respectively.

We comment only very briefly here on the SO(8) model, details of which are given elsewhere (Barnes *et al* 1978). It suffices to say that the quarks are assigned to the fundamental representation 8_+, which has the SU(4)×U(1) decomposition $8_+ = \bar{4}_{-\frac{1}{2}} + 4_{\frac{1}{2}}$. Outer products of 8_+ give successively

$$
\begin{aligned}
&8_+ \times 8_+ = 28_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} + (35_+ + 1)_{\square\square},\\
&8_+ \times 8_+ \times 8_+ = 56_{-\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}} + 2(160_+ + 8_+)_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} + (112_+ + 8_+)_{\square\square\square},
\end{aligned}
$$

the symmetry being indicated by the Young frame. Mesons are assigned to the multiplets $28 + 35_+ + 1$ (for both spin-0 and spin-1), and baryons to the multiplets $160_+ + 8_+$ (for spin-$\frac{1}{2}$) and $112_+ + 8_+$ (for spin-$\frac{3}{2}$). The SU(4)×U(1) decompositions of the meson multiplets, with which we are concerned in the following, are

$$28 = 6_{-1} + (15 + 1)_0 + 6_1, \qquad 35_+ = \overline{10}_{-1} + 15_0 + 10_1, \qquad \text{and} \qquad 1 = 1_0.$$

3. Computation of the SU(4) singlet factors

In order to compute SO(8) Clebsch–Gordan coefficients, we require the matrix

† In particular, the labels X', X'', are redundant for the following SO(8) representations: $[\sigma, \sigma, \sigma, \sigma']$, $[\sigma, \sigma, \sigma', \sigma']$, $[\sigma, \sigma', \sigma', \sigma']$ and $[\sigma, \sigma, \frac{1}{2}, -\frac{1}{2}]$.

Table 1. Labels and dimensions of SO(8) multiplets and their $SU(4)\times U(1)_X$ decompositions.

$[\sigma]$	Dynkin label	Dimension and $SU(4)\times U(1)_X$ restriction
$[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$	$\begin{pmatrix}0 & 0 & {}^{1}_{0}\end{pmatrix}$	$8_+=\bar{4}_{-\frac{1}{2}}+4_{+\frac{1}{2}}$
$[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}]$	$\begin{pmatrix}0 & 0 & {}^{0}_{1}\end{pmatrix}$	$8_-=4_{-\frac{1}{2}}+\bar{4}_{+\frac{1}{2}}$
$[1\ 0\ 0\ 0]$	$\begin{pmatrix}1 & 0 & {}^{0}_{0}\end{pmatrix}$	$8=1_{-1}+6_0+1_1$
$[1\ 1\ 0\ 0]$	$\begin{pmatrix}0 & 1 & {}^{0}_{0}\end{pmatrix}$	$28=6_{-1}+15_0+1_0+6_1$
$[1\ 1\ 1\ 1]$	$\begin{pmatrix}0 & 0 & {}^{2}_{0}\end{pmatrix}$	$35_+=\overline{10}_{-1}+15_0+10_1$
$[1\ 1\ 1\ -1]$	$\begin{pmatrix}0 & 0 & {}^{0}_{2}\end{pmatrix}$	$35_-=10_{-1}+15_0+\overline{10}_1$
$[2\ 0\ 0\ 0]$	$\begin{pmatrix}2 & 0 & {}^{0}_{0}\end{pmatrix}$	$35=1_{-2}+6_{-1}+20''_0+1_0+6_1+1_2$
$[\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}]$	$\begin{pmatrix}0 & 0 & {}^{3}_{0}\end{pmatrix}$	$112_+=\overline{20}_{-3/2}+\overline{36}_{-\frac{1}{2}}+36_{+\frac{1}{2}}+20_{+\frac{3}{2}}$
$[\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ -\frac{3}{2}]$	$\begin{pmatrix}0 & 0 & {}^{0}_{3}\end{pmatrix}$	$112_-=20_{-\frac{3}{2}}+36_{-\frac{1}{2}}+\overline{36}_{+\frac{1}{2}}+\overline{20}_{+\frac{3}{2}}$
$[3\ 0\ 0\ 0]$	$\begin{pmatrix}3 & 0 & {}^{0}_{0}\end{pmatrix}$	$112=1_{\pm 3}+6_{\pm 2}+(20''+1)_{\pm 1}+(50+6)_0$
$[\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}]$	$\begin{pmatrix}0 & 1 & {}^{1}_{0}\end{pmatrix}$	$160_+=\overline{20}'_{-\frac{3}{2}}+(\overline{36}+20'+\bar{4})_{-\frac{1}{2}}+(36+\overline{20}'+4)_{\frac{1}{2}}+20'_{\frac{3}{2}}$
$[\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2}\ -\frac{1}{2}]$	$\begin{pmatrix}0 & 1 & {}^{0}_{1}\end{pmatrix}$	$160_-=20'_{-\frac{3}{2}}+(36+\overline{20}'+4)_{-\frac{1}{2}}+(\overline{36}+20'+\bar{4})_{\frac{1}{2}}+\overline{20}'_{\frac{3}{2}}$
$[2\ 1\ 0\ 0]$	$\begin{pmatrix}1 & 1 & {}^{0}_{0}\end{pmatrix}$	$160=6_{\pm 2}+(20''+15+1)_{\pm 1}+(64+6+6)_0$

elements of the SO(8) generators in any irreducible representation. These are known in terms of the Gel'fand basis, equation (2.5) (Gel'fand *et al* 1963, Gilmore 1970). For example,

$$\langle\sigma^+|\Sigma_{2c+1\,2c}|\sigma\rangle = \left(\frac{\prod_{a=1}^{c}(\tau_{a\,2c+1}+\tau_{b\,2c})(\tau_{a\,2c+1}-\tau_{b\,2c}-1)\prod_{a=1}^{c-1}(\tau_{a\,2c-1}+\tau_{b\,2c})(\tau_{a\,2c-1}-\tau_{b\,2c-1})}{4\prod_{a=1,a\neq b}^{c}(\tau_{a\,2c}^2-\tau_{b\,2c}^2)[\tau_{a\,2c}^2-(\tau_{b\,2c}+1)^2]}\right)^{1/2}$$

where

$$\sigma^+_{aA}=\sigma_{aA}+\delta_{ab}\delta_{A2c}, \qquad 1\leqslant b\leqslant c,$$

$$\tau_{a\,2c}=\sigma_{a\,2c}+c-a \quad \text{and} \quad \tau_{a\,2c+1}=\sigma_{a\,2c+1}+c-a+1. \tag{3.1}$$

However, as emphasised above, we wish here to work in the SU(4) × U(1) basis, and to use previous SU(4) results (Haacke *et al* 1976).

Firstly, we establish the connection between the orthogonal (SO(6) ⊃ SO(5) ⊃ ... ⊃ SO(2)) and unitary (SU(4) ⊃ SU(3) ⊃ ... ⊃ U(1)) Gel'fand bases. The representation labels $[\sigma]$ and $\{\nu\}$ of SO(6) and SU(4) are related through

$$\begin{aligned}\sigma_1 &= \tfrac{1}{2}(\nu_1+\nu_2-\nu_3)\\ \sigma_2 &= \tfrac{1}{2}(\nu_1-\nu_2+\nu_3)\\ \sigma_3 &= \tfrac{1}{2}(\nu_1-\nu_2-\nu_3).\end{aligned} \tag{3.2}$$

Using equation (2.3), we can identify the isospin subgroup $SU(2)_I$ of SU(4) with one of the SU(2) subgroups of $SO(4) \cong SU(2) \times SU(2)/Z_2 \times Z_2$ in SO(6), generated by $\Sigma_{ab} = -\Sigma_{ba}$, $a, b = 1, 2, 3, 4$. The generators are:

$$\begin{aligned}I_+ &= A_{12} = D_{12} = \tfrac{1}{2}(-\Sigma_{23}+\mathrm{i}\,\Sigma_{31}) - \tfrac{1}{2}(-\Sigma_{14}+\mathrm{i}\,\Sigma_{24}),\\ I_3 &= \tfrac{1}{2}(H_1-H_2) = \tfrac{1}{2}(K_1-K_2) = \tfrac{1}{2}\mathrm{i}\,(\Sigma_{12}-\Sigma_{34}).\end{aligned} \tag{3.3}$$

The other SU(2) factor has the generators:

$$\begin{aligned}W_+ &= -A_{\bar{3}\bar{4}} = -D_{1\bar{2}} = \tfrac{1}{2}(-\Sigma_{23}+\mathrm{i}\,\Sigma_{31}) + \tfrac{1}{2}(-\Sigma_{14}+\mathrm{i}\,\Sigma_{24}),\\ W_3 &= \tfrac{1}{2}(-H_3+H_4) = \tfrac{1}{2}(K_1+K_2) = \tfrac{1}{2}\,\mathrm{i}(\Sigma_{12}+\Sigma_{34})\end{aligned} \tag{3.4}$$

and we shall distinguish it as $SU(2)_W$. The representation labels (W, I) of $SU(2)_W \times SU(2)_I$ and $[\sigma_{14}, \sigma_{24}]$ of SO(4) are related by

$$W = \tfrac{1}{2}(\sigma_{14}+\sigma_{24}) \qquad I = \tfrac{1}{2}(\sigma_{14}-\sigma_{24}). \tag{3.5}$$

Now the state of highest weight $|\Omega\rangle$ in any irreducible representation $\{\nu_1\ \nu_2\ \nu_3\}$ of SU(4) is unique (up to a phase). In the Gel'fand basis

$$|\Omega\rangle = \left|\begin{matrix}\nu_1 & & \nu_2 & & \nu_3 & & 0\\ & \nu_1 & & \nu_2 & & \nu_3 & \\ & & \nu_1 & & \nu_2 & & \\ & & & \nu_1 & & & \end{matrix}\right\rangle$$

and in terms of the labels I_3, Y, and Z (equation (2.8)), since (Gel'fand *et al* 1963)

$$H_a\left|\begin{matrix}\nu_{14} & & \nu_{24} & & \nu_{34} & & \nu_{44}\\ & \nu_{13} & & \nu_{23} & & \nu_{33} & \\ & & \nu_{12} & & \nu_{22} & & \\ & & & \nu_{11} & & & \end{matrix}\right\rangle$$

$$= \left(\sum_{b=1}^{a}\nu_{ba} - \sum_{b=1}^{a-1}\nu_{b\,a-1} - \tfrac{1}{4}\sum_{b=1}^{4}\nu_{b4}\right)\left|\begin{matrix}\nu_{14} & & \nu_{24} & & \nu_{34} & & \nu_{44}\\ & \nu_{13} & & \nu_{23} & & \nu_{33} & \\ & & \nu_{12} & & \nu_{22} & & \\ & & & \nu_{11} & & & \end{matrix}\right\rangle,$$

we have (Haacke *et al* 1976)

$$
\begin{aligned}
I_3|\Omega\rangle &= \tfrac{1}{2}(\nu_1-\nu_2)|\Omega\rangle \\
Y|\Omega\rangle &= \tfrac{1}{3}(\nu_1+\nu_2-2\nu_3)|\Omega\rangle \\
Z|\Omega\rangle &= \tfrac{1}{4}(\nu_1+\nu_2+\nu_3)|\Omega\rangle .
\end{aligned}
\tag{3.6}
$$

We can use these conditions to identify the corresponding state $|\Omega\rangle'$ in the $\mathrm{SO}(6)\supset \mathrm{SO}(5)\ldots\supset\mathrm{SO}(2)$ basis. In view of equations (2.3) and (2.8),

$$
K_1|\Omega\rangle'=\sigma_3|\Omega\rangle' \qquad K_2|\Omega\rangle'=-\sigma_2|\Omega\rangle' \qquad K_3|\Omega\rangle'=-\sigma_1|\Omega\rangle' . \tag{3.7}
$$

From equations (3.3) and (3.4), it is evident that $K_1=\frac{1}{2}(W_3+I_3)$ is already diagonal in the $\mathrm{SO}(6)\supset\mathrm{SO}(5)\supset\mathrm{SO}(4)\supset\ldots\supset\mathrm{SO}(2)$ basis, whereas $K_2=\frac{1}{2}(W_3-I_3)$ may be diagonalised in addition by passing to the $\mathrm{SU}(2)_W\times\mathrm{SU}(2)_I$ basis for SO(4); the basis transformation coefficients are just SU(2) Clebsch–Gordan coefficients. Finally, $K_3=\mathrm{i}\,\Sigma_{56}$ must be diagonalised by computing its matrix elements in this basis (cf equation (3.1)). The existence of $|\Omega\rangle$ ensures that there is a unique solution $|\Omega\rangle'$ to equations (3.7), up to a phase†.

Once this identification has been established, the basis transformation between $\mathrm{SU}(4)\supset\mathrm{SU}(3)\supset\ldots\supset\mathrm{U}(1)$ states and $\mathrm{SO}(6)\supset\mathrm{SO}(5)\supset\ldots\supset\mathrm{SO}(2)$ states is completed by acting with the appropriate lowering operators, $A^\dagger_{12}=D^\dagger_{12}$, $A^\dagger_{23}=D^\dagger_{23}$ and $A^\dagger_{34}=D^\dagger_{\bar{1}2}$, on $|\Omega\rangle=|\Omega\rangle'$, the matrix elements being computed using results like equation (3.1) (Gel'fand *et al* 1963, Gilmore 1970, Haacke *et al* 1976). This procedure is illustrated in table 2 for the $\underset{\sim}{15}$ (i.e. {2 1 1} of SU(4), and [1 1 0] of SO(6)). The notation for labelling SU(4) states is explained in § 4.

The next stage in establishing the matrix elements of SO(8) generators in the $\mathrm{SU}(4)\times\mathrm{U}(1)$ basis is to diagonalise the U(1) generator $X=K_4=\mathrm{i}\,\Sigma_{78}$. This is readily done and requires taking linear combinations

$$
\frac{1}{\sqrt{2}}|1\rangle\pm\frac{\mathrm{i}}{\sqrt{2}}|2\rangle
$$

of appropriate pairs of Gel'fand states, corresponding to a transformation from an $\mathrm{SO}(6)\times\mathrm{SO}(2)$ basis to the $\mathrm{SO}(6)\times\mathrm{U}(1)\cong\mathrm{SU}(4)/Z_2\times\mathrm{U}(1)$ basis. However these linear combinations are defined only up to an overall phase for each $\mathrm{SU}(4)\times\mathrm{U}(1)$ multiplet. This phase arbitrariness is reflected, for example, in the matrix elements of the generator Σ_{76} in this basis, and must be removed by a choice of phase convention.

At this point it should be recalled that the adoption of the $\mathrm{SU}(4)\times\mathrm{U}(1)$ basis involves the Gel'fand phase convention for SU(4) that the matrix elements of A_{12}, A_{23} and A_{34} (or, in view of equations (2.3), of D_{12}, D_{23} and $D_{\bar{1}2}$), be positive. Also, from equations (2.2), we have

$$
\Sigma_{76}=\tfrac{1}{2}(D_{34}-D^\dagger_{34}+D_{\bar{3}4}-D^\dagger_{\bar{3}4}). \tag{3.8}
$$

† Another possibility is to start instead with the state of highest weight in the SO(6) basis, and to identify the corresponding state in the SU(4) basis. This can be done explicitly, and we have, up to a phase,

$$
\left|\begin{array}{ccccc}
\sigma_1 & & \sigma_2 & & \sigma_3 \\
 & \sigma_1 & & \sigma_2 & \\
 & \sigma_1 & & \sigma_2 & \\
 & & \sigma_1 & & \\
 & & \sigma_1 & &
\end{array}\right\rangle
=
\left|\begin{array}{ccccccc}
\nu_1 & & \nu_2 & & \nu_3 & & 0 \\
 & \nu_2 & & \nu_3 & & 0 & \\
 & & \nu_2 & & \nu_3 & & \\
 & & & \nu_2 & & &
\end{array}\right\rangle
$$

Table 2. Basis transformation between SU(4) and SO(6) basis states for $\underset{\sim}{15}$.

$\lvert\nu_{ij}\rangle^{\{211\}}$	$\left\lvert\begin{array}{ccc}\mu&&Z\\\lambda&&Y\\&I_3&\end{array}\right\rangle^{\{211\}}$	$\lvert\sigma_{iI}\rangle^{[110]}$
$\left\lvert\begin{array}{ccccc}2&&1&&1\\&2&&1&\\&&2&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}3&&1\\2&&\frac{1}{3}\\&\frac{1}{2}&\end{array}\right\rangle$	$-\frac{1}{2}\left\lvert\begin{array}{ccc}1&&1\\1&&0\\&1&\\&0&\end{array}\right\rangle+\frac{i}{2}\left\lvert\begin{array}{ccc}1&&1\\1&&0\\&0&\\&0&\end{array}\right\rangle+\frac{i}{2}\left\lvert\begin{array}{ccc}1&&0\\1&&0\\&1&\\&0&\end{array}\right\rangle+\frac{1}{2}\left\lvert\begin{array}{ccc}1&&0\\1&&0\\&0&\\&0&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}2&&1&&1\\&2&&1&\\&&1&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}3&&1\\2&&\frac{1}{3}\\&-\frac{1}{2}&\end{array}\right\rangle$	$-\frac{1}{\sqrt{2}}\left\lvert\begin{array}{ccc}1&&1\\1&&0\\&1&\\&-1&\end{array}\right\rangle+\frac{i}{\sqrt{2}}\left\lvert\begin{array}{ccc}1&&0\\1&&0\\&1&\\&-1&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}2&&1&&1\\&1&&1&\\&&1&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}3&&1\\1&&-\frac{2}{3}\\&0&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}1&&1\\1&&1\\&1&\\&-1&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}2&&1&&0\\&2&&1&\\&&2&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}8&&0\\2&&1\\&\frac{1}{2}&\end{array}\right\rangle$	$\frac{1}{\sqrt{2}}\left\lvert\begin{array}{ccc}1&&1\\1&&0\\&1&\\&1&\end{array}\right\rangle-\frac{i}{\sqrt{2}}\left\lvert\begin{array}{ccc}1&&0\\1&&0\\&1&\\&1&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}2&&1&&0\\&2&&1&\\&&1&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}8&&0\\2&&1\\&-\frac{1}{2}&\end{array}\right\rangle$	$\frac{1}{2}\left\lvert\begin{array}{ccc}1&&1\\1&&0\\&1&\\&0&\end{array}\right\rangle+\frac{i}{2}\left\lvert\begin{array}{ccc}1&&1\\1&&0\\&0&\\&0&\end{array}\right\rangle-\frac{i}{2}\left\lvert\begin{array}{ccc}1&&0\\1&&0\\&1&\\&0&\end{array}\right\rangle+\frac{1}{2}\left\lvert\begin{array}{ccc}1&&0\\1&&0\\&0&\\&0&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}2&&1&&0\\&2&&0&\\&&2&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}8&&0\\3&&0\\&1&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}1&&1\\1&&-1\\&1&\\&1&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}2&&1&&0\\&2&&0&\\&&1&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}8&&0\\3&&0\\&0&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}1&&1\\1&&-1\\&1&\\&0&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}2&&1&&0\\&1&&1&\\&&1&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}8&&0\\1&&0\\&0&\end{array}\right\rangle$	$-\frac{1}{\sqrt{3}}\left\lvert\begin{array}{ccc}1&&1\\1&&1\\&1&\\&0&\end{array}\right\rangle+\sqrt{\frac{2}{3}}\left\lvert\begin{array}{ccc}1&&0\\0&&0\\&0&\\&0&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}1&&1&&1\\&1&&1&\\&&1&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}1&&0\\1&&0\\&0&\end{array}\right\rangle$	$-\sqrt{\frac{2}{3}}\left\lvert\begin{array}{ccc}1&&1\\1&&1\\&1&\\&0&\end{array}\right\rangle-\frac{1}{\sqrt{3}}\left\lvert\begin{array}{ccc}1&&0\\0&&0\\&0&\\&0&\end{array}\right\rangle$
$\left\lvert\begin{array}{ccccc}2&&1&&0\\&2&&0&\\&&0&&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}8&&0\\3&&0\\&-1&\end{array}\right\rangle$	$\left\lvert\begin{array}{ccc}1&&1\\1&&-1\\&1&\\&-1&\end{array}\right\rangle$

Table 2—*continued*

$\lvert\nu_{ij}\rangle^{\{211\}}$	$\left\lvert\begin{array}{cc}\mu & Z\\ \lambda & Y\\ \multicolumn{2}{c}{I_3}\end{array}\right\rangle^{\{211\}}$	$\lvert\sigma_{iI}\rangle^{[110]}$
$\left\lvert\begin{array}{ccc}2&1&0\\ \multicolumn{3}{c}{1\quad 0}\\ \multicolumn{3}{c}{1}\end{array}\right\rangle$	$\left\lvert\begin{array}{cc}8&0\\ 2&-1\\ \multicolumn{2}{c}{\frac{1}{2}}\end{array}\right\rangle$	$-\frac{1}{2}\left\lvert\begin{array}{cc}1&0\\1&0\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle+\frac{i}{2}\left\lvert\begin{array}{cc}1&1\\1&0\\ \multicolumn{2}{c}{0}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle-\frac{i}{2}\left\lvert\begin{array}{cc}1&0\\1&0\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle-\frac{1}{2}\left\lvert\begin{array}{cc}1&0\\1&0\\ \multicolumn{2}{c}{0}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle$
$\left\lvert\begin{array}{ccc}2&1&0\\ \multicolumn{3}{c}{1\quad 0}\\ \multicolumn{3}{c}{0}\end{array}\right\rangle$	$\left\lvert\begin{array}{cc}8&0\\ 2&-1\\ \multicolumn{2}{c}{-\frac{1}{2}}\end{array}\right\rangle$	$-\frac{1}{\sqrt{2}}\left\lvert\begin{array}{cc}1.&1\\1&0\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{-1}\end{array}\right\rangle-\frac{i}{\sqrt{2}}\left\lvert\begin{array}{cc}1&0\\1&0\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{-1}\end{array}\right\rangle$
$\left\lvert\begin{array}{ccc}1&1&0\\ \multicolumn{3}{c}{1\quad 1}\\ \multicolumn{3}{c}{1}\end{array}\right\rangle$	$\left\lvert\begin{array}{cc}\bar{3}&-1\\ 1&\frac{2}{3}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle$	$\left\lvert\begin{array}{cc}1&1\\1&1\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{1}\end{array}\right\rangle$
$\left\lvert\begin{array}{ccc}1&1&0\\ \multicolumn{3}{c}{1\quad 0}\\ \multicolumn{3}{c}{1}\end{array}\right\rangle$	$\left\lvert\begin{array}{cc}\bar{3}&-1\\ 2&-\frac{1}{3}\\ \multicolumn{2}{c}{\frac{1}{2}}\end{array}\right\rangle$	$\frac{1}{\sqrt{2}}\left\lvert\begin{array}{cc}1&1\\1&0\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{1}\end{array}\right\rangle+\frac{i}{\sqrt{2}}\left\lvert\begin{array}{cc}1&0\\1&0\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{1}\end{array}\right\rangle$
$\left\lvert\begin{array}{ccc}1&1&0\\ \multicolumn{3}{c}{1\quad 0}\\ \multicolumn{3}{c}{0}\end{array}\right\rangle$	$\left\lvert\begin{array}{cc}\bar{3}&-1\\ 2&-\frac{1}{3}\\ \multicolumn{2}{c}{-\frac{1}{2}}\end{array}\right\rangle$	$\frac{1}{2}\left\lvert\begin{array}{cc}1&1\\1&0\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle+\frac{i}{2}\left\lvert\begin{array}{cc}1&1\\1&0\\ \multicolumn{2}{c}{0}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle+\frac{i}{2}\left\lvert\begin{array}{cc}1&0\\1&0\\ \multicolumn{2}{c}{1}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle-\frac{1}{2}\left\lvert\begin{array}{cc}1&0\\1&0\\ \multicolumn{2}{c}{0}\\ \multicolumn{2}{c}{0}\end{array}\right\rangle$

Noting equations (2.8), the shift properties $(\Delta X, \Delta Z, \Delta Y)$ of D_{34}, $D_{\bar{3}4}$ are $(-1, -\frac{1}{2}, -\frac{2}{3})$ and $(-1, \frac{1}{2}, \frac{2}{3})$, respectively. However,

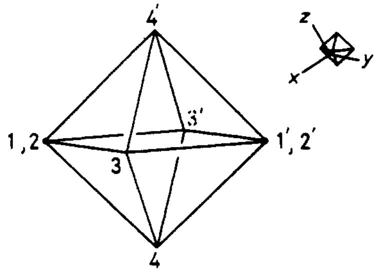
$$[D_{\bar{2}3}, [D_{23}, D_{34}]] = -D_{\bar{3}4}$$

so that, in addition to D_{12}, D_{23} and $D_{\bar{1}2}$, only one of D_{34} and $D_{\bar{3}4}$ is an independent shifting operator.

In analogy with the SU(4) case, we adopt the convention that the matrix elements of D_{34} be positive; where a given state is mapped to two different states in the same $SU(4) \times U(1)$ multiplet, the state with higher weight has a positive matrix element. (In the unitary case, this qualification is superfluous, since all matrix elements are positive.)

With this choice, the transformation between the $SO(7) \supset SO(6)$ and the $SU(4) \times U(1)$ bases in any irreducible representation of SO(8) is determined, and the matrix elements of the SO(8) generators in the $SU(4) \times U(1)$ basis may be written down, enabling direct products of irreducible representations of SO(8) to be decomposed into their irreducible representations of SO(8) to be decomposed into their irreducible parts. Table 3 gives the matrix elements of D_{34} in the basic representation $\underset{\sim}{8}_{+}$. The notation for labelling states is explained in § 4.

Table 3. Non-zero matrix elements of D_{34} in the $\underset{\sim}{8}_+$ multiplet.



$$\langle 1'|D_{34}|1\rangle = 1$$
$$\langle 2'|D_{34}|2\rangle = 1$$

4. Tables of singlet factors

The general SO(8) Clebsch–Gordan coefficient is written as

$$\mathrm{CG} = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \nu X \end{pmatrix}\begin{pmatrix} \nu_1 & \nu_2 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}\begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y \end{pmatrix} C^{\lambda_1\lambda_2}_{I_{3_1}I_{3_2}I_3}$$

where the irreducible representation labels $[\sigma_1\,\sigma_2\,\sigma_3\,\sigma_4]$, $\{\nu_1\,\nu_2\,\nu_3\}$, $\{\mu_1\,\mu_2\}$ and $\{\lambda_1\}$ of SO(8), SU(4), SU(3) and SU(2), respectively, have been replaced by their dimensions σ, ν, μ and λ, and

$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \nu X \end{pmatrix}$ is the SU(4) singlet factor,

$\begin{pmatrix} \nu_1 & \nu_2 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$ is the SU(3) singlet factor,

$\begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y \end{pmatrix}$ is the SU(2) (de Swart) factor,

and $C^{\lambda_1\lambda_2\lambda}_{I_{3_1}I_{3_2}I_3}$ is the SU(2) Clebsch–Gordan coefficient.

Singlet factors are arranged in tables, according to the values of ν and X (or μ and Z, λ and Y, etc). The following phase conventions for the singlet factors are adopted.

(i) The highest Clebsch–Gordan coefficient in any expansion is defined to be $+1$. This means that the highest single factor is also $+1$.

(ii) For fixed σ (or ν, μ, etc), the highest singlet factor in the highest table is chosen to be positive. The highest table is the one having the highest X, and then highest ν (or highest Z, and then highest μ, etc). Within a table, the highest singlet factor is the one having the highest ν_1, and then highest ν_2 (or highest μ_1, and then highest μ_2, etc). $\underset{\sim}{15}_D$ is taken to be higher than $\underset{\sim}{15}_F$, and $\underset{\sim}{8}_D$ higher than $\underset{\sim}{8}_F$.

(iii) 'Highest' for irreducible representations $\{\nu\}$ of SU(4) means that irreducible representation having maximal Z, then maximal Y, then maximal I_3. These definitions carry over similarly to SU(3), and SU(2). In the latter case, 'highest' simply means 'highest dimensional'.

Table 4. SU(3) singlet factors for $\nu_1 \times \nu_2$ contains ν, with ν_1, ν_2, ν any of $\underset{\sim}{1}$, $\underset{\sim}{4}$, $\underset{\sim}{\bar{4}}$, $\underset{\sim}{6}$, $\underset{\sim}{10}$, $\underset{\sim}{\overline{10}}$ and $\underset{\sim}{15}$ of SU(4).

A	$4 \times 4 = 6 + 10$
B	$4 \times \bar{4} = 1 + 15$
C	$4 \times 6 = \bar{4} + 20'$
D	$4 \times \overline{10} = \bar{4} + \overline{36}$
E	$4 \times 15 = 4 + 20' + 36$
F	$6 \times 6 = 1 + 15 + 20''$
G	$6 \times 10 = 15 + 45$
H	$6 \times 15 = 6 + 10 + \overline{10} + 64$
I	$10 \times \overline{10} = 1 + 15 + 84$
J	$10 \times 15 = 6 + 10 + 64 + 70$

A

$$\begin{pmatrix} 4 & 4 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$$

	η_1	η_3
6	−	+
10	+	+

$6\ \frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	10
3	$\frac{1}{4}$	3	$\frac{1}{4}$	1

$\bar{3}\ \frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	6
3	$\frac{1}{4}$	3	$\frac{1}{4}$	1

$3\ -\frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	6	10
3	$\frac{1}{4}$	1	$-\frac{3}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$
1	$-\frac{3}{4}$	3	$\frac{1}{4}$	$-1/\sqrt{2}$	$1/\sqrt{2}$

$1\ -\frac{3}{2}$

μ_1	Z_1	μ_2	Z_2	10
1	$-\frac{3}{4}$	1	$-\frac{3}{4}$	

B

$$\begin{pmatrix} 4 & \bar{4} & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$$

	η_1	η_3
1	−	−
15	+	+

$\bar{3}\ -1$

μ_1	Z_1	μ_2	Z_2	15
1	$-\frac{3}{4}$	$\bar{3}$	$-\frac{1}{4}$	1

$8\ 0$

μ_1	Z_1	μ_2	Z_2	15
3	$\frac{1}{4}$	$\bar{3}$	$-\frac{1}{4}$	1

$1\ 0$

μ_1	Z_1	μ_2	Z_2	1	15
3	$\frac{1}{4}$	$\bar{3}$	$-\frac{1}{4}$	$\sqrt{3}/2$	$\frac{1}{2}$
1	$-\frac{3}{4}$	1	$\frac{3}{4}$	$-\frac{1}{2}$	$\sqrt{3}/2$

$3\ 1$

μ_1	Z_1	μ_2	Z_2	15
3	$\frac{1}{4}$	1	$\frac{3}{4}$	1

Table 4.—*continued*

C

$$\begin{pmatrix} 4 & 6 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$$

	η_1	η_3
$\bar{4}$	$-$	$+$

$1\ \frac{3}{4}$

μ_1	Z_1	μ_2	Z_2	$\bar{4}$
3	$\frac{1}{4}$	$\bar{3}$	$\frac{1}{2}$	$+1$

$\bar{3}\ -\frac{1}{4}$

μ_1	Z_1	μ_2	Z_2	$\bar{4}$
3	$\frac{1}{4}$	3	$-\frac{1}{2}$	$+\sqrt{2/3}$
1	$-\frac{3}{4}$	$\bar{3}$	$\frac{1}{2}$	$+1/\sqrt{3}$

D

$$\begin{pmatrix} 4 & \overline{10} & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$$

	η_1	η_3
$\bar{4}$	$-$	$-$

$1\ \frac{3}{4}$

μ_1	Z_1	μ_2	Z_2	$\bar{4}$
3	$\frac{1}{4}$	$\bar{3}$	$\frac{1}{2}$	$+\sqrt{3/5}$
1	$-\frac{3}{4}$	1	$\frac{3}{2}$	$-\sqrt{2/5}$

$\bar{3}\ -\frac{1}{4}$

μ_1	Z_1	μ_2	Z_2	$\bar{4}$
3	$\frac{1}{4}$	$\bar{6}$	$-\frac{1}{2}$	$+2/\sqrt{5}$
1	$-\frac{3}{4}$	$\bar{3}$	$\frac{1}{2}$	$-1/\sqrt{5}$

E

$$\begin{pmatrix} 4 & 15 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$$

	η_1	η_3
4	$-$	$-$

$1\ -\frac{3}{4}$

μ_1	Z_1	μ_2	Z_2	4
1	$-\frac{3}{4}$	1	0	$-1/\sqrt{5}$
3	$\frac{1}{4}$	$\bar{3}$	-1	$+2/\sqrt{5}$

$3\ \frac{1}{4}$

μ_1	Z_1	μ_2	Z_2	4
1	$-\frac{3}{4}$	3	1	$-2/\sqrt{15}$
3	$\frac{1}{4}$	8	0	$+\frac{4}{3}\sqrt{2/5}$
3	$\frac{1}{4}$	1	0	$+1/3\sqrt{5}$

F

$$\begin{pmatrix} 6 & 6 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$$

	η_1	η_3
1	$+$	$+$
15	$-$	$-$
20″	$+$	$+$

$\bar{6}\ 1$

μ_1	Z_1	μ_2	Z_2	20″
$\bar{3}$	$\frac{1}{2}$	$\bar{3}$	$\frac{1}{2}$	$+1$

$3\ 1$

μ_1	Z_1	μ_2	Z_2	15
$\bar{3}$	$\frac{1}{2}$	$\bar{3}$	$\frac{1}{2}$	$+1$

$1\ 0$

μ_1	Z_1	μ_2	Z_2	1	15
$\bar{3}$	$\frac{1}{2}$	3	$-\frac{1}{2}$	$+1/\sqrt{2}$	$-1/\sqrt{2}$
3	$-\frac{1}{2}$	$\bar{3}$	$\frac{1}{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$

$8\ 0$

μ_1	Z_1	μ_2	Z_2	15	20″
$\bar{3}$	$\frac{1}{2}$	3	$-\frac{1}{2}$	$-1/\sqrt{2}$	$+1/\sqrt{2}$
3	$-\frac{1}{2}$	$\bar{3}$	$\frac{1}{2}$	$+1/\sqrt{2}$	$+1/\sqrt{2}$

$\bar{3}\ -1$

μ_1	Z_1	μ_2	Z_2	15
3	$-\frac{1}{2}$	3	$-\frac{1}{2}$	-1

$6\ -1$

μ_1	Z_1	μ_2	Z_2	20″
3	$-\frac{1}{2}$	3	$-\frac{1}{2}$	$+1$

G

$\begin{pmatrix} 6 & 10 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$

	η_1	η_3
15	−	+

$3\ 1$

μ_1	Z_1	μ_2	Z_2	15
$\bar{3}$	$\frac{1}{2}$	6	$\frac{1}{2}$	$+1$

$1\ 0$

μ_1	Z_1	μ_2	Z_2	15
$\bar{3}$	$\frac{1}{2}$	3	$-\frac{1}{2}$	$+1$

$8\ 0$

μ_1	Z_1	μ_2	Z_2	15
$\bar{3}$	$\frac{1}{2}$	3	$-\frac{1}{2}$	$-\frac{1}{2}$
3	$-\frac{1}{2}$	6	$\frac{1}{2}$	$+\sqrt{3}/2$

$3\ -1$

μ_1	Z_1	μ_2	Z_2	15
$\bar{3}$	$\frac{1}{2}$	1	$-\frac{3}{2}$	$-1/\sqrt{2}$
3	$-\frac{1}{2}$	3	$-\frac{1}{2}$	$+1/\sqrt{2}$

H

$\begin{pmatrix} 6 & 15 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & u Z \end{pmatrix}$

	η_1	η_3
6	−	−
10	−	+
$\overline{10}$	−	+

$1\ \frac{3}{2}$

μ_1	Z_1	μ_2	Z_2	$\overline{10}$
$\bar{3}$	$\frac{1}{2}$	3	1	$+1$

$6\ \frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	10
$\bar{3}$	$\frac{1}{2}$	8	0	$+\sqrt{2/3}$
3	$-\frac{1}{2}$	3	1	$-1/\sqrt{3}$

$\bar{3}\ \frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	6	$\overline{10}$
3	$-\frac{1}{2}$	3	1	$-\sqrt{2/5}$	$+1/\sqrt{3}$
$\bar{3}$	$\frac{1}{2}$	8	0	$+2\sqrt{2/15}$	$+\frac{2}{3}$
$\bar{3}$	$\frac{1}{2}$	1	0	$+1/\sqrt{15}$	$-\sqrt{2}/3$

$3\ -\frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	6	10
3	$-\frac{1}{2}$	8	0	$+2\sqrt{2/15}$	$-\frac{2}{3}$
3	$-\frac{1}{2}$	1	0	$-1/\sqrt{15}$	$-\sqrt{2}/3$
$\bar{3}$	$\frac{1}{2}$	$\bar{3}$	-1	$\sqrt{2/5}$	$+1/\sqrt{3}$

$\bar{6}\ -\frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	$\overline{10}$
3	$-\frac{1}{2}$	8	0	$+\sqrt{2/3}$
$\bar{3}$	$\frac{1}{2}$	$\bar{3}$	-1	$-1/\sqrt{3}$

$1\ -\frac{3}{2}$

μ_1	Z_1	μ_2	Z_2	10
3	$-\frac{1}{2}$	$\bar{3}$	-1	-1

Table 4—*continued*

I

$\begin{pmatrix} 10 & \overline{10} & \vert & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \vert & \mu Z \end{pmatrix}$

	η_1	η_3
1	+	+
15	−	−

$\bar{3}\ -1$

μ_1	Z_1	μ_2	Z_2	15
3	$-\frac{1}{2}$	$\bar{6}$	$-\frac{1}{2}$	$+\sqrt{2/3}$
1	$-\frac{3}{2}$	$\bar{3}$	$\frac{1}{2}$	$-1/\sqrt{3}$

$1\ 0$

μ_1	Z_1	μ_2	Z_2	1	15
6	$\frac{1}{2}$	$\bar{6}$	$-\frac{1}{2}$	$+\sqrt{3/5}$	$+1/\sqrt{3}$
3	$-\frac{1}{2}$	$\bar{3}$	$\frac{1}{2}$	$-\sqrt{3/10}$	$+1/\sqrt{6}$
1	$-\frac{3}{2}$	1	$\frac{3}{2}$	$+1/\sqrt{10}$	$-1/\sqrt{2}$

$8\ 0$

μ_1	Z_1	μ_2	Z_2	15
6	$\frac{1}{2}$	$\bar{6}$	$-\frac{1}{2}$	$+\sqrt{5/6}$
3	$-\frac{1}{2}$	$\bar{3}$	$\frac{1}{2}$	$-1/\sqrt{6}$

$3\ 1$

μ_1	Z_1	μ_2	Z_2	15
3	$-\frac{1}{2}$	1	$\frac{3}{2}$	$-1/\sqrt{3}$
6	$\frac{1}{2}$	$\bar{3}$	$\frac{1}{2}$	$+\sqrt{2/3}$

J

$\begin{pmatrix} 10 & 15 & \vert & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \vert & \mu Z \end{pmatrix}$

	η_1	η_3
6	+	−
10	−	−

$\bar{3}\ \frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	6
3	$-\frac{1}{2}$	3	1	$-1/\sqrt{5}$
6	$\frac{1}{2}$	8	0	$+2/\sqrt{5}$

$6\ \frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	10
6	$\frac{1}{2}$	1	0	$+1/3\sqrt{3}$
6	$\frac{1}{2}$	8	0	$+2\sqrt{5}/3\sqrt{3}$
3	$-\frac{1}{2}$	3	1	$-\sqrt{2}/3$

$3\ -\frac{1}{2}$

μ_1	Z_1	μ_2	Z_2	6	10
1	$-\frac{3}{2}$	3	1	$+1/\sqrt{5}$	$-\sqrt{2}/3$
3	$-\frac{1}{2}$	1	0	$-\sqrt{2/15}$	$-1/3\sqrt{3}$
3	$-\frac{1}{2}$	8	0	$-2/\sqrt{15}$	$+2\sqrt{2}/3\sqrt{3}$
6	$\frac{1}{2}$	$\bar{3}$	−1	$+\sqrt{2/5}$	$+\frac{2}{3}$

$1\ -\frac{3}{2}$

μ_1	Z_1	μ_2	Z_2	10
1	$-\frac{3}{2}$	1	0	$-1/\sqrt{3}$
3	$-\frac{1}{2}$	$\bar{3}$	−1	$+\sqrt{2/3}$

Table 5. SU(4) singlet factors for $\sigma_1 \times \sigma_2$ contains σ, with $\sigma_1, \sigma_2, \sigma$ any of $\underset{\sim}{1}$, $\underset{\sim}{8}_+$, $\underset{\sim}{28}$ and $\underset{\sim}{35}_+$ of SO(8).

A	$8_+ \times 8_+$	$= 1 + 28 + 35_+$
B	$8_+ \times 28$	$= 8_+ + 56_- + 160_+$
C	$8_+ \times 35_+$	$= 8_+ + 160_+ + 112_+$
D	28×28	$= 1 + 28 + 35_+ + 35_- + 35 + 300 + 350$
E	$28 \times 35_+$	$= 28 + 35_+ + 350 + 567_+$
F	$35_+ \times 35_+$	$= 1 + 28 + 35_+ + 300 + 567_+ + 294_+$

A

$$\left(\begin{array}{cc|c} 8_+ & 8_+ & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \nu X \end{array}\right)$$

	ξ_1	ξ_3
1	+	+
28	−	−
35_+	+	+

$6\ -1$

$\nu_1\ X_1$	$\nu_2\ X_2$	28
$\bar{4}\ -\frac{1}{2}$	$\bar{4}\ -\frac{1}{2}$	-1

$\overline{10}\ -1$

$\nu_1\ X_1$	$\nu_2\ X_2$	35_+
$\bar{4}\ -\frac{1}{2}$	$\bar{4}\ -\frac{1}{2}$	$+1$

$15\ 0$

$\nu_1\ X_1$	$\nu_2\ X_2$	28	35_+
$4\ \frac{1}{2}$	$\bar{4}\ -\frac{1}{2}$	$-1/\sqrt{2}$	$+1/\sqrt{2}$
$\bar{4}\ -\frac{1}{2}$	$4\ \frac{1}{2}$	$+1/\sqrt{2}$	$+1/\sqrt{2}$

$1\ 0$

$\nu_1\ X_1$	$\nu_2\ X_2$	1	28
$4\ \frac{1}{2}$	$\bar{4}\ -\frac{1}{2}$	$-1/\sqrt{2}$	$+1/\sqrt{2}$
$\bar{4}\ -\frac{1}{2}$	$4\ \frac{1}{2}$	$+1/\sqrt{2}$	$+1/\sqrt{2}$

$6\ 1$

$\nu_1\ X_1$	$\nu_2\ X_2$	28
$4\ \frac{1}{2}$	$4\ \frac{1}{2}$	$+1$

$10\ 1$

$\nu_1\ X_1$	$\nu_2\ X_2$	35_+
$4\ \frac{1}{2}$	$4\ \frac{1}{2}$	$+1$

B

$$\left(\begin{array}{cc|c} 8_+ & 28 & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \nu X \end{array}\right)$$

	ξ_1	ξ_3
8_+	−	−

$\bar{4}\ -\frac{1}{2}$

$\nu_1\ X_1$	$\nu_2\ X_2$	8_+
$\bar{4}\ -\frac{1}{2}$	$15\ 0$	$+\frac{1}{2}\sqrt{15/7}$
$4\ \frac{1}{2}$	$6\ -1$	$-\sqrt{3/7}$
$\bar{4}\ -\frac{1}{2}$	$1\ 0$	$+1/2\sqrt{7}$

$4\ \frac{1}{2}$

$\nu_1\ X_1$	$\nu_2\ X_2$	8_+
$4\ \frac{1}{2}$	$15\ 0$	$+\frac{1}{2}\sqrt{15/7}$
$\bar{4}\ -\frac{1}{2}$	$6\ 1$	$+\sqrt{3/7}$
$4\ \frac{1}{2}$	$1\ 0$	$-1/2\sqrt{7}$

Table 5.—*continued*

C

$\begin{pmatrix} 8_+ & 35_+ & \vert & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \vert & \nu X \end{pmatrix}$

	ξ_1	ξ_3
8_+	+	+

$\bar{4}\ -\frac{1}{2}$

ν_1	X_1	ν_2	X_2	8_+
$\bar{4}$	$-\frac{1}{2}$	15	0	$+\sqrt{3/7}$
4	$\frac{1}{2}$	$\overline{10}$	−1	$-2/\sqrt{7}$

$4\ \frac{1}{2}$

ν_1	X_1	ν_2	X_2	8_+
4	$\frac{1}{2}$	15	0	$-\sqrt{3/7}$
$\bar{4}$	$-\frac{1}{2}$	10	1	$+2/\sqrt{7}$

D

$\begin{pmatrix} 28 & 28 & \vert & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \vert & \nu X \end{pmatrix}$

	ξ_1	ξ_3
1	+	+
28	−	−
35_+	+	+

$6\ -1$

ν_1	X_1	ν_2	X_2	28
1	0	6	−1	$-1/2\sqrt{3}$
15	0	6	−1	$\frac{1}{2}\sqrt{5/3}$
6	−1	1	0	$+1/2\sqrt{3}$
6	−1	15	0	$\frac{1}{2}\sqrt{5/3}$

$\overline{10}\ -1$

ν_1	X_1	ν_2	X_2	35_+
15	0	6	−1	$+1/\sqrt{2}$
6	−1	15	0	$-1/\sqrt{2}$

$1\ 0$

ν_1	X_1	ν_2	X_2	1	28
6	1	6	−1	$-\sqrt{3/14}$	$1/\sqrt{2}$
1	0	1	0	$-1/2\sqrt{7}$	0
15	0	15	0	$+\frac{1}{2}\sqrt{15/7}$	0
6	−1	6	1	$-\sqrt{3/14}$	$-1/\sqrt{2}$

$15\ 0$

ν_1	X_1	ν_2	X_2	28	35_+	
6	1	6	−1	$+1/\sqrt{6}$	$+1/\sqrt{6}$	
15	0	1	0	0	$+1/2\sqrt{3}$	
15	0	15	0	0	$+1/\sqrt{2}$	*D*
15	0	15	0	$+\sqrt{2/3}$	0	*F*
1	0	15	0	0	$+1/2\sqrt{3}$	
6	−1	6	1	$+1/\sqrt{6}$	$-1/\sqrt{6}$	

$6\ 1$

ν_1	X_1	ν_2	X_2	28
6	1	1	0	$-1/2\sqrt{3}$
6	1	15	0	$\frac{1}{2}\sqrt{5/3}$
15	0	6	1	$\frac{1}{2}\sqrt{5/3}$
1	0	6	1	$-1/2\sqrt{3}$

$10\ 1$

ν_1	X_1	ν_2	X_2	35_+
6	1	15	0	$-1/\sqrt{2}$
15	0	6	1	$+1/\sqrt{2}$

E

$$\begin{pmatrix} 28 & 35_+ & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \nu X \end{pmatrix}$$

	ξ_1	ξ_3
28	−	+
35_+	−	−

6 −1

ν_1	X_1	ν_2	X_2	28
15	0	$\overline{10}$	−1	$-\sqrt{2/3}$
6	−1	15	0	$-1/\sqrt{3}$

$\overline{10}$ −1

ν_1	X_1	ν_2	X_2	35_+
15	0	$\overline{10}$	−1	$+\frac{3}{4}$
1	0	$\overline{10}$	−1	$-\frac{1}{4}$
6	−1	15	0	$-\frac{1}{2}\sqrt{3/2}$

15 0

ν_1	X_1	ν_2	X_2	28	35_+
6	1	$\overline{10}$	−1	$-2/\sqrt{15}$	$+\frac{1}{2}$
15	0	15	0	$-\sqrt{2/5}$	0 *D*
15	0	15	0	0	$+1/\sqrt{2}$ *F*
1	0	15	0	$-1/\sqrt{15}$	0
6	−1	10	1	$-2/\sqrt{15}$	$-\frac{1}{2}$

1 0

ν_1	X_1	ν_2	X_2	28
15	0	15	0	−1

6 1

ν_1	X_1	ν_2	X_2	28
15	0	10	1	$+\sqrt{2/3}$
6	1	15	0	$+1/\sqrt{3}$

10 1

ν_1	X_1	ν_2	X_2	35_+
15	0	10	1	$+\frac{3}{4}$
1	0	10	1	$+\frac{1}{4}$
6	1	15	0	$+\frac{1}{2}\sqrt{3/2}$

F

$$\begin{pmatrix} 35_+ & 35_+ & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \nu X \end{pmatrix}$$

	ξ_1	ξ_3
1	+	+
28	−	−
35_+	+	+

6 −1

ν_1	X_1	ν_2	X_2	28
15	0	$\overline{10}$	−1	$+1/\sqrt{2}$
$\overline{10}$	−1	15	0	$-1/\sqrt{2}$

$\overline{10}$ −1

ν_1	X_1	ν_2	X_2	35_+
15	0	$\overline{10}$	−1	$-1/\sqrt{2}$
$\overline{10}$	−1	15	0	$+1/\sqrt{2}$

15 0

ν_1	X_1	ν_2	X_2	28	35_+
10	1	$\overline{10}$	−1	$-\sqrt{3/10}$	$+1/\sqrt{3}$
15	0	15	0	0	$-1/\sqrt{3}$ *D*
15	0	15	0	$\sqrt{2/5}$	0 *F*
$\overline{10}$	−1	10	1	$-\sqrt{3/10}$	$-1/\sqrt{3}$

1 0

ν_1	X_1	ν_2	X_2	1	28
10	1	$\overline{10}$	−1	$+1/\sqrt{5}$	$-1/\sqrt{2}$
15	0	15	0	$-\sqrt{3/5}$	0
$\overline{10}$	−1	10	1	$+1/\sqrt{5}$	$+1/\sqrt{2}$

6 1

ν_1	X_1	ν_2	X_2	28
15	0	10	1	$+1/\sqrt{2}$
10	1	15	0	$-1/\sqrt{2}$

10 1

ν_1	X_1	ν_2	X_2	35_+
15	0	10	1	$+1/\sqrt{2}$
10	1	15	0	$-1/\sqrt{2}$

The singlet factors have the following symmetry properties, characterised by real phase factors, under interchange of factors 1 and 2, and complex conjugation:

$$\left(\begin{array}{cc|c} \sigma_1 & \sigma_2 & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \nu \end{array}\right) = \xi_1\eta_1 \left(\begin{array}{cc|c} \sigma_2 & \sigma_1 & \sigma \\ \nu_2 X_2 & \nu_1 X_1 & \nu X \end{array}\right)$$

$$\left(\begin{array}{cc|c} \sigma_1 & \sigma_2 & \sigma \\ \nu_1 X_1 & \nu_2 X_2 & \nu X \end{array}\right) = \xi_3\eta_3 \left(\begin{array}{cc|c} \sigma_1 & \sigma_2 & \sigma \\ \bar{\nu}_1 - X_1 & \bar{\nu}_2 - X_2 & \bar{\nu} - X \end{array}\right)$$

$$\left(\begin{array}{cc|c} \nu_1 & \nu_2 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{array}\right) = \eta_1\epsilon_1 \left(\begin{array}{cc|c} \nu_2 & \nu_1 & \nu \\ \mu_2 Z_2 & \mu_1 Z_1 & \mu Z \end{array}\right)$$

$$\left(\begin{array}{cc|c} \nu_1 & \nu_2 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu z \end{array}\right) = \eta_3\epsilon_3 \left(\begin{array}{cc|c} \bar{\nu}_1 & \bar{\nu}_2 & \bar{\nu} \\ \bar{\mu}_1 - Z_1 & \bar{\mu}_2 - Z_2 & \bar{\mu} - Z \end{array}\right)$$

$$\left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y \end{array}\right) = \epsilon_1(-1)^{I-I_1-I_2} \left(\begin{array}{cc|c} \mu_2 & \mu & \mu \\ \lambda_2 Y_2 & \lambda_1 Y_1 & \lambda Y \end{array}\right)$$

$$\left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y \end{array}\right) = \epsilon_3(-1)^{I-I_1-I_2} \left(\begin{array}{cc|c} \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu} \\ \lambda_1 - Y_1 & \lambda_2 - Y_2 & \lambda - Y \end{array}\right)$$

$$C^{I_1 I_2 I}_{I_{3_1} I_{3_2} I_3} = (-1)^{I-I_1-I_2} C^{I_2 I_1 I}_{I_{3_2} I_{3_1} I_3}$$

$$C^{I_1 I_2 I}_{I_{3_1} I_{3_2} I_3} = (-1)^{I-I_1-I_2} C^{I_1 I_2 I}_{-I_{3_1} -I_{3_2} -I_3}$$

These relations embody the fact that for SO(8) and SU(2), the complex conjugates $[\bar{\sigma}]$ and $\{\bar{\lambda}\}$ are equivalent to $[\sigma]$ and $\{\lambda\}$, respectively.

Table 4 gives the relevant SU(3) singlet factors required, in addition to those given by Haacke *et al* (1976), where the same phase conventions are used. All the relevant SU(2) singlet factors are also to be found there.

Table 5 gives the SU(4) singlet factors for all SO(8) decompositions involving the representations $\underset{\sim}{1}$, $\underset{\sim}{8}_+$, $\underset{\sim}{28}$ and $\underset{\sim}{35}_+$ sufficient to perform calculations of the mass breaking, and of scattering and decay amplitudes in the meson sector of the SO(8) model. The symmetry factors ξ_1 and ξ_3 are also given.

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